

satisfies  $T^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ .

Thm 3 (The closed graph theorem)

Let  $T: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ . Then

$T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2) \iff$

$$\Gamma(T) = \{(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2 \mid y = Tx\}$$

is closed in  $\mathcal{B}_1 \oplus \mathcal{B}_2$  (a closed subspace).

Corollary (The Hellinger-Toeplitz theorem) Let  $A: \mathcal{H} \rightarrow \mathcal{H}$  be s.t.

$$(x, Ay) = (Ax, y) \quad \forall x, y \in \mathcal{H}.$$

Then  $A$  is bounded (and self-adjoint,  $A = A^*$ ).

Proof  $\Gamma(A)$  is closed.

### I. 3 Locally convex vector spaces

Def A vector space  $V$  is a linear topological space if it is a topological space (Hausdorff) and vector space operations are continuous.

Def A l.t.s. is metrizable if it admits a metric space structure s.t. topology defined by metric is the same as the original topology.

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Def A complex topological vector space  $V$  (t.v.s) is locally convex (l.c.) if every open  $U \ni 0$  contains an absolutely convex absorbing open  $U_1 \ni 0$ , i.e.

- $x, y \in U_1$  and  $\alpha, \beta \in \mathbb{C}$ ,  $|\alpha| + |\beta| = 1$   
 $\Rightarrow \alpha x + \beta y \in U_1$
- $\forall x \in V \exists t > 0$  s.t.  $tx \in U_1$ .

Def A seminorm  $p: V \rightarrow \mathbb{R}_{\geq 0}$  is map satisfying

- (i)  $p(x+y) \leq p(x) + p(y)$
- (ii)  $p(\lambda x) = |\lambda| p(x)$ ,  $\lambda \in \mathbb{C}$

Note:  $p(x) = 0$  does not necessarily implies  $x = 0$ .

Theorem | A t.v.s.  $V$  is locally convex, if and only if it has a family  $\{p_\lambda\}_{\lambda \in \Lambda}$  of separating seminorms:

$$p_\lambda(x) = 0 \quad \forall \lambda \in \Lambda \Rightarrow x = 0.$$

The topology on  $V$  coincides with the natural topology: the weakest topology in which all  $p_\lambda$ ,  $\lambda \in \Lambda$ , are continuous.

Proof If part is easy:  $\{x \in V\}$

$p_V(x) < \varepsilon\}$  are absolutely convex absorbing open sets in  $V$ .

Conversely, if  $V$  is locally convex, then Minkowski function

$$p_V(x) = \inf_{\alpha > 0, \bar{x}' x \in V} \alpha$$

of an absolutely convex absorbing  $V \subset M$  is seminorm. Indeed,  $\forall \varepsilon > 0$

$$\frac{x}{p_V(x) + \varepsilon} \in V, \quad \frac{y}{p_V(y) + \varepsilon} \in V,$$

so that

$$\frac{x+y}{p_V(x) + p_V(y) + 2\varepsilon} = \frac{p_V(x) + \varepsilon}{p_V(x) + p_V(y) + 2\varepsilon}$$

$$\cdot \frac{x}{p_V(x) + \varepsilon} + \frac{p_V(y) + \varepsilon}{p_V(x) + p_V(y) + 2\varepsilon} \cdot \frac{y}{p_V(y) + \varepsilon} \in V.$$

$p(\alpha x) = p(x)$ ,  $|\alpha| = 1$  because  
 $\alpha V = V$  (absolutely convex).

$p(tx) = t p(x)$ ,  $t > 0$  because  $V$  is absorbing.

Theorem 2 Let  $V$  be l.c. space.  
Then the following are equivalent:

- (a)  $V$  is metrizable
- (b)  $0 \in V$  has a countable neighborhood base
- (c) The topology on  $V$  is generated by a countable system of seminorms.

Proof (a)  $\Rightarrow$  (b) - general  
(b)  $\Rightarrow$  (c) - from Thm 1.

Need to prove (c)  $\Rightarrow$  (a). Direct construction: if  $\{p_n\}_{n=1}^{\infty}$  is the family of seminorms generating the topology, then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{p_n(x-y)}{1 + p_n(x-y)} 2^{-n}$$

is a metric.

Proof Each term satisfies triangle inequality:  $\frac{t}{1+t} \nearrow$  on  $(0, \infty)$

and

$$\frac{\alpha}{1+\alpha+\beta} \leq \frac{\alpha}{1+\alpha}, \text{ etc.}$$

$$\frac{\alpha+\beta}{1+\alpha+\beta} \leq \frac{\alpha}{1+\alpha} + \frac{\beta}{1+\beta}, \quad \alpha, \beta \geq 0.$$

$d(x, y) = 0 \Leftrightarrow x = y$  (separating property)

[because  $\frac{a+c}{b+d} \leq \frac{a}{b} + \frac{c}{d}$  ( $a, b, c, d > 0$ )]

Def A Fréchet space is metrizable,  
complete l.c. space

All Banach thms: uniform boundedness principle, open and inverse mapping theorems, hold for Fréchet spaces.  
closed graph theorem  
(Was proved by Banach).

Example 1  $C^\infty(\Omega)$ ,  $\Omega \subset \mathbb{R}^N$ ,

domain, is a Fréchet space.

(Topology on  $C^\infty(\Omega)$  is defined by sequential convergence:

$f_n \rightarrow f$  in  $C^\infty(\Omega)$ , if

$D^\alpha f_n \Rightarrow D^\alpha f$  on every compact  $K \subset \Omega$

$\forall \alpha = (\alpha_1, \dots, \alpha_N)$ , where

$$D^\alpha = \overset{\infty}{\underset{1}{\partial}} \dots \overset{\infty}{\underset{N}{\partial}} \alpha_i, \quad \partial_i = \frac{\partial}{\partial x_i}.$$

Let  $\bigcup_{m=0} K_m = \Omega$  be the exhaustion of  $\Omega$ :

$K_m \subset K_{m+1} \quad \forall m, \quad K_m - \text{compact.}$

Set

$$P_{m,n}(f) = \sum_{|\alpha| \leq n} \max_{x \in K_m} |D^\alpha f(x)|, \quad n \in \mathbb{Z}_{\geq 0}$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_N$ .

Note: The family of seminorms

$p_n = p_{n,\alpha}$ ,  $n \in \mathbb{Z}_{\geq 0}$  is sufficient  
(i.e. defines the same topology). Exercise

Metric topology on  $C^\infty(\Omega)$  implies  
sequential convergence and  $C^\infty(\Omega)$   
is closed under it.

Ex 2  $\mathcal{S}(\mathbb{R}^N)$  - the Schwartz space  
of rapidly decreasing functions:

$$\mathcal{S}(\mathbb{R}^N) = \{ f \in C^\infty(\mathbb{R}^N) \mid$$

$$\|f\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta f(x)| < \infty \quad \forall \alpha, \beta \}$$

is a Fréchet space with the natural  
topology given by the seminorms  $p_{\alpha, \beta} = \| \cdot \|_{\alpha, \beta}$

Indeed, it remains to show that  $\mathcal{S}(\mathbb{R}^N)$   
is complete. Let  $\{f_n\}$  be a Cauchy  
sequence. Since  $C(\mathbb{R}^N)$  is complete,

$x^\alpha D^\beta f_n \Rightarrow g_{\alpha, \beta}$ . We need to show  
that  $g = g_{0,0} \in C^\infty(\mathbb{R}^N)$  and

$g_{\alpha, \beta} = x^\alpha D^\beta g$ . But it follows  
from the following fact

$f_n \in C^1(\mathbb{R}^N)$ ,  $f_n \Rightarrow f$  and

$\partial_i f_n \Rightarrow g_i$  on  $\mathbb{R}^N$ , then  $f \in C^1(\mathbb{R}^N)$  &  
 $g_i = \partial_i f$ ,

by induction.

(Say

$$f_n(x_1, \dots, x_N) = f_n(0, x_2, \dots, x_N) + \int_0^{x_1} \partial_1 f_n(t, x_2, \dots, x_N) dt$$

$\partial_1 f_n \Rightarrow g_1$  so that can pass to a limit  
under the integral, i.e.

$$f(\vec{x}) = f(0, x_2, \dots, x_N) + \int_0^{x_1} g_1(t, x_2, \dots, x_N) dt$$

Directed family of seminorms:  $\forall \alpha, \beta$

$\exists C > 0$  &  $\gamma \in A$  s.t.

$$p_\alpha(x) + p_\beta(x) \leq C p_\gamma(x).$$

Always can do; for metrizable l.c. space  
one can always get

$$p_1 \leq p_2 \leq p_3 \leq \dots$$

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(to start Lecture 7)

Theorem Let  $V_1, V_2$  be metrizable  
l.c. vector spaces with seminorms  
 $p_n & q_n$  respectively ( $p_1 \leq p_2 \leq \dots$  &  
 $q_1 \leq q_2 \leq \dots$ ). Then a linear map

$$T: V_1 \rightarrow V_2$$

is continuous if and only if  $\forall m$   
 $\exists n$  s.t.

$$q_m(Tx) \leq C p_n(x) \quad \forall x \in V_1.$$

Corollary  $T \in V^*$  (topological  
dual) if and only if

$$\|T(x)\| \leq C p_n(x) \quad \forall x \in V$$

and some  $n$ .

Now we return to tempered  
distributions  $\mathcal{S}'(\mathbb{R}^N)$  (topology on it  
- later)

$$p_{n,m}(f) = \|f\|_{n,m} = \sum \max_{\substack{x \in \mathbb{R}^N \\ |\alpha| \leq n}} |x^\alpha D^\beta f(x)|.$$

$$|\beta| \leq m$$

$$\|x^\alpha D^\beta f(x)\|.$$

- $T \in \mathcal{S}'(\mathbb{R}^N) \Leftrightarrow \{T(f)\} \leq C \|f\|_{n,m}$   
 $\forall f \in \mathcal{S}(\mathbb{R}^N)$  and some  $m, n$ .

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Ex 1 "Dirac  $\delta$ -function"  $\forall x_0 \in \mathbb{R}^N$

$$\delta_{x_0}(f) = f(x_0) (= \int_{\mathbb{R}^N} f(x) \delta(x-x_0) dx)$$

$$\|\delta_{x_0}(f)\| \leq \|f\|_{0,0}$$

Ex 2  $g \in \mathcal{S}(\mathbb{R}^N)$ ,

$$g(f) = \int_{\mathbb{R}^N} g(x) f(x) dx$$

$$\|g(f)\| \leq \|g\|_{L^1} \|f\|_{0,0}$$

$$\mathcal{S}(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N).$$

Ex 3  $T \in \mathcal{S}'(\mathbb{R}^N)$ ,

$$(D^\alpha T)(f) = (-1)^{|\alpha|} T(D^\alpha f),$$

$$D^\alpha T \in \mathcal{S}'(\mathbb{R}^N)$$

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\frac{dH}{dx} = \delta_0 \quad \left( \frac{dH}{dx}(f) = - \int_0^\infty f'(x) dx = f(0) \right).$$

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Ex 4 v.p.  $\frac{1}{x}$  =  $\mathcal{P} \frac{1}{x}$

$$\mathcal{P}\left(\frac{1}{x}\right)(f) = \lim_{\epsilon \rightarrow 0^+} \int_{|x| \geq \epsilon} \frac{f(x)}{x} dx$$

$$= \int_0^\infty \frac{f(x) - f(-x)}{x} dx < \infty$$

$$\left\{ \frac{f(x) - f(-x)}{x} \right\} \leq \frac{1}{x} \int_{-x}^x |f'(t)| dt \\ \leq 2 \|f\|_{0,1} \quad \text{and}$$

$$\left\{ \mathcal{P}\left(\frac{1}{x}\right)(f) \right\} \leq 2 \|f\|_{0,1} + 2 \left| \int_1^\infty \frac{x f(x)}{x^2} dx \right| \\ \leq 2 (\|f\|_{0,1} + \|f\|_{1,0}).$$

Lecture 8

9/19/03

Ex 5 Measures of polynomially bounded growth:

$$\mu(|x| \leq R) < C R^M$$

$$\left\{ \int_R^\infty f(x) d\mu(x) \right\} - \underline{\text{exercise}}$$

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- $(V_a T)(f) = T(V_{-a} f)$

$$(V_a f)(x) = f(x-a), \quad a \in \mathbb{R}^N$$

$$A : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \det A \neq 0$$

$$(V(A)f)(x) = f(A^{-1}x)$$

- $(V(A)T)(f) = |\det A| \cdot T(V(\bar{A}^t)f)$

$$\underline{\delta\left(\frac{x}{a}\right)} = |a| \delta(x)$$

Let  $C_{PB}^\infty(\mathbb{R}^N)$  be polynomially bounded smooth functions on  $\mathbb{R}^N$ .

Thm (regularity theorem for distributions) Let  $T \in \mathcal{S}'(\mathbb{R}^N)$ .

Then  $\exists g \in C_{PB}^0(\mathbb{R}^N)$  s.t  
 $\underbrace{\quad}_{\text{& } \underline{\alpha}}$

$$\underline{T = D^\alpha g}$$

Schwartz kernel theorem Let

$B : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m) \rightarrow \mathbb{C}$ ,  
separately continuous. Then  $\exists$   
 $T \in \mathcal{S}'(\mathbb{R}^{n+m})$  s.t.  $B(f, g) = T(f \otimes g)$

(Will prove this and other results later, after doing Fourier transform a Hermite functions).

Ex (a) Let  $(M, \mu)$  be a measure space,  $\mu(M) < \infty$  and let  $K \in L^\infty(M \times M)$ . Then

$$(\hat{K}f)(x) = \int_M K(x, y) f(y) d\mu(y)$$

is a bounded linear operator in  $L^2(M, d\mu)$  (an integral operator with the kernel  $K(x, y)$ ).

$$\|\hat{K}f\|^2 = \int_M \left| \int_M K(x, y) f(y) d\mu(y) \right|^2 d\mu(x)$$

$$\leq \int_M \int_M |K(x, y)|^2 d\mu(y) \|f\|^2 d\mu(x)$$

$$\leq \int_{M \times M} |K(x, y)|^2 d\mu(x) d\mu(y) \|f\|^2.$$

$M \times M$

In general, if  $(M, \mu)$  is any measure space, then if  $K(x, y)$  is

s.t.  $K \in L^2(M \times M, dy \times dy)$ , then  
 $\hat{K}$  is bounded integral operator with  
 the kernel  $K(x, y)$ , called Hilbert  
 - Schmidt operator;

$$\|\hat{K}\| \leq \|K\|$$

$L^2(M \times M, dy \times dy)$

(b)  $L^2(\mathbb{R})$  identity operator  $I$   
 is not integral, but by the Schwartz  
 theorem it has a kernel

$$\delta(x-y) = \delta(y-x)$$

$$f(x) = \int_{\mathbb{R}} \delta(y-x) f(y) dy.$$

$\mathbb{R}$

(c) If  $D = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$

is a differential operator in  $L^2(\mathbb{R}^N)$   
 (say with domain  $\mathcal{S}(\mathbb{R}^N)$ ), then its  
 distributional kernel is

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_y^\alpha (a_\alpha(y) \delta(y-x))$$

$|\alpha| \leq m$

$$= \sum_{|\alpha| \leq m} b_\alpha(y) D_y^\alpha \delta(y-x).$$

$C_0^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$

Ex 3  $C_0^\infty(\mathbb{R}^N)$  (or  $C_0^\infty(\Omega)$ ) is not a Fréchet space, but is an inductive limit of Fréchet spaces.

Namely, let

$$C_0^\infty(K) = \left\{ f \in C_0^\infty(\mathbb{R}^N) \mid \text{supp}(f) \subseteq K \right\}$$

$K \subset \mathbb{R}^N$  - compact. Then  $C_0^\infty(K)$  is a Fréchet space with the system of seminorms

$$P_{m,K}(f) = \max_{x \in K} \left| D^\alpha f(x) \right|, \quad |\alpha| \leq m$$

Let

$$\mathbb{R}^N = \bigcup_{n=0}^{\infty} K_n, \quad K_n \subset K_{n+1}.$$

so that

$$C_0^\infty(\mathbb{R}^N) = \bigcup_{n=0}^{\infty} C_0^\infty(K_n).$$

Def Let  $F_n$  be l.c. vector spaces s.t.  $F_n \subseteq F_{n+1}$  and restriction of the topology of  $F_{n+1}$  onto  $F_n$  coincides with topology of  $F_n$ . Then

$$F = \varinjlim F_n = \lim_{\text{ind}} F_n$$

is a l.c. vector space - strict inductive

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limit of l.c. vector spaces  $F_n$ , defined by

$$F = \bigcup_{n=0}^{\infty} F_n$$

with a l.c. topology with a base of 0 given by absolutely convex absorbing  $V$  s.t.  $V \cap F_n$  is open for all  $n$ .

Theorem Let  $F = \varinjlim F_n$ . Then

I) (a)  $F$  is l.c. vector space

(b) Topology on is the strongest l.c. topology s.t. all maps

$$F_n \hookrightarrow F$$

are continuous.

(c) The restriction of the topology on  $F$  to  $F_n$  is the given topology on  $F_n$ .

(d) If each  $F_n$  is complete then  $F$  is complete.

II)  $T: F \rightarrow V$ ,  $V$  - l.c. vector space.  
The linear map  $T$  is continuous iff each  $T|_{F_n}$  is continuous.

III)  $f_n \rightarrow f$  in  $F$  iff  $\exists N$  s.t. all  $f_n \in F_N$  and  $f_n \rightarrow f$  in  $F_N$ .

(For this we need  $F_n \not\subseteq F_{n+1}$ ,  $F_n$  is closed in  $F_{n+1}$ )

The proof is based on the following lemma.

Lemma Let  $V$  be a l.c. vector space and  $V_1 \subset V$  be its subspace with relative topology (automatically l.c.). Then if  $U_1 \subset V_1$  is open, absolutely convex, then

$\exists U \subset V, \quad \text{---} \parallel \quad \text{---} \parallel \text{---}, \text{ s.t.}$

$$U_1 = U \cap V_1.$$

Proof  $\exists O \subset V$ , open, s.t.

$$U_1 = O \cap V_1;$$

$O \supset O_1$  - absolutely convex & absorbing.

Set

$$U = U \{ \alpha x + \beta O_1 \}.$$

$$x \in U_1$$

$$|\alpha| + |\beta| = 1$$

Then  $U$  is open, absolutely convex & absorbing.  
Clearly,

$$U \cap V_1 \supset U_1.$$

However, if  $\alpha x + \beta y \in U \cap V_1$ ,

$x \in U_1, y \in O_1$ , then  $y \in O_1 \cap V \subset U$ .

Since  $U_1$  is absolutely convex,  $\alpha x + \beta y \in U_1$ .

Thus

$$U \cap V_1 = U_1.$$

Proof of (d). Suppose  $F$  is not complete, let  $\overline{F}$  be its completion;  $\exists f \in \overline{F}$  s.t.  $x \notin F_n$  for all  $n$ .

Every  $F_n \subset \overline{F}$  is closed,  $\exists U_n \subset \overline{F}$ , a.c., s.t.

$$(x + U_n) \cap F_n = \emptyset;$$

can choose  $U_1 \supseteq U_2 \supseteq \dots$ ; let

$U$  be the a.c. hull of

$$\bigcup_{n=1}^{\infty} (\frac{1}{2} U_n \cap F_n),$$

$U \subset F$ , open  $\Rightarrow \overline{U} \subset \overline{F}$ , open. Then

$$(x + \overline{U}) \cap F \neq \emptyset \Rightarrow \exists n \text{ s.t. } (x + \overline{U}) \cap F_n \neq \emptyset$$

( $F$  is dense in  $\overline{F}$ ).

Claim: For all  $n$ ,  $\overline{U} \subset U_n + F_n$ .

This would be a contradiction, since

$$\emptyset \neq (x + \overline{U}) \cap F_n \subset (x + U_n + F_n)$$

$$\cap F_n = (x + U_n) \cap F_n + F_n$$

$$= \emptyset + F_n = \emptyset.$$

Proof of the claim.

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$$\bar{U} \subset U + \frac{1}{2}U_n,$$

$$y \in \bar{U}, \quad y = \sum_r^l \lambda_r x_r, \quad x_r \in (\frac{1}{2}U_n) \cap \mathcal{F}_n.$$

$$\sum_r^l |\lambda_r| = 1.$$

But

$$\sum_{r \leq n}^l \lambda_r x_r \in \mathcal{F}_n; \quad \sum_{r > n}^l \lambda_r x_r \in$$

$$\sum_{r > n}^l \lambda_r \cdot \frac{1}{2}U_r \subset \frac{1}{2}U_n, \text{ so}$$

$$U \subset \frac{1}{2}U_n + \mathcal{F}_n, \quad \bar{U} \subseteq U_n + \mathcal{F}_n.$$

Proof of I(a): sufficient to prove I(c).

Let  $U_n \subset F_n$ , open,  $U_n \neq \emptyset$ . By lemma,  
 $\exists V_{n+1} \subset F_{n+1}$ , open, s.t.  $V_{n+1} \cap F_n = U_n$ .

Set

$$\mathcal{O} = \bigcup_{m=n}^{\infty} V_m.$$

Clearly,  $\mathcal{O} \subset F$  - open, since  $\mathcal{O} \cap F_K$   
open in  $F_K, \forall K$ ;  $\mathcal{O} \cap F_n = U_n \Rightarrow$   
relative  $F/F_n$  topology = topology on  $F_n$ .

I(b) - easy, I(d) - exercise (see also the proof  
of III).

Now II): if  $T|_{F_n}$  is continuous, let  
 $U \subset V$ -open (ab.c.&a); then

$$T^{-1}(U) \cap F_n = (T|_{F_n})^{-1}(U)$$

- open in  $F_n, \forall n$ . Since  $T^{-1}(U)$  is a.c.ka.,  
it is open in  $F$  by definition.

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Finally, proof of III). By contradiction:  
suppose  $\forall n \exists f_{m_n} \notin F_n$ . Set

$$V_1 = F_1, \tilde{f}_1 = f_{m_1} \in F_{n_2} = V_2,$$

$\tilde{f}_2 = f_{m_{n_2}} \in F_{n_3} = V_3$ , etc. So we  
have a subsequence  $\{\tilde{f}_n\}$  s.t.

$$\tilde{f}_n \in V_{n+1} \setminus V_n.$$

By induction, since  $V_n \subset F$  is closed,  
 $f_n \notin V_n$ ,  $\exists V_n \ni f_n$ , a.c.l.a.,

$V_n \cap V_n \neq \emptyset$ , separating seminorm  $p_n \Rightarrow$   
 $\exists \ell_n \in F^*$  (by Hahn-Banach) s.t.

$$\ell_n|_{V_n} = 0, \quad \ell_n(\tilde{f}_n) = \text{"given"} \\ = n - \sum_{k=1}^{n-1} \ell_k(\tilde{f}_n).$$

Set  $\ell = \sum_{n=1}^{\infty} \ell_n$ ; since  $\ell|_{F_n}$  is  
 a finite sum,  $\ell \in F^*$ . Since  $\tilde{f}_n \xrightarrow{n} f$   
 in  $F$ ,  $\ell(\tilde{f}_n) \rightarrow \ell(f)$ . But

$$\ell(\tilde{f}_n) = n - \text{a contradiction!}$$

— . — . — . —

Def A distribution (generalized function)  
 is a continuous linear functional on  $\mathcal{D}(\Omega)$ .  
 The space of all distributions is  $\mathcal{D}'(\Omega)$ .

Basic fact  $T \in \mathcal{D}'(\Omega)$  iff  $\forall K \subset \Omega$

$\exists C > 0$  &  $n$  s.t.

$$|T(f)| \leq C \sum_{|\alpha| \leq n} \max_{x \in K} |D^\alpha f(x)|$$

$\forall f \in C_0^\infty(K)$ .

Examples

1.  $g \in C(\Omega)$ ,  $T = D^\alpha g \in \mathcal{D}'(\Omega)$ ,

$$T(f) = (-1)^\alpha \int_{\Omega} g(x) D^\alpha f(x) dx;$$

$\forall K \subset \Omega$ ,  $f \in C_0^\infty(K)$

$$|T(f)| \leq C \|f\|_{0,n} \max_{x \in K} |g(x)|.$$

Strong & weak solutions

$$D = \sum_{|\alpha| \leq n} a_\alpha(x) D^\alpha,$$

$$(DT)(f) = T \left( \sum_{|\alpha| \leq n} (-1)^\alpha D^\alpha (a_\alpha f) \right).$$

$Df = u$  - strong solution of PDE,  $f \in C^\infty$

$DT = u$  - weak  $\| \cdot \|$ ,  $T \in \mathcal{D}'$

If  $f \in C^K$  and  $u \in C_0^\infty$ , then  
 $f$  is a weak sol  $\Leftrightarrow f$  is a strong sol.

Def 1 A l.c. vector space  $V$  is barrelled if every a.c. & a and closed set (= barrel) contains an open set  $\ni 0$ . (I.e., every barrel is a neighborhood.)

Def 2  $E \subset V$ , ( $V$ -l.c.v. space),  $E$  is bounded, if it is absorbed by every neighborhood of origin.

Def 3 A barrelled (<sup>always</sup> Hausdorff) vector space  $V$  is Montel space, if all closed bounded sets of  $V$  are compact.

Facts

(holomorphic functions  
on  $\Omega \subseteq \mathbb{C}$ )

• Spaces  $\{S(\mathbb{R}^n), H(\Omega), C_0^\infty(\Omega)\}$   
(Fréchet)

are Montel spaces.

- The dual of a Montel space with the strong topology\* is Montel.
- A normed space in a Montel space iff it is finite-dimensional.

~~weak\*~~  
(Strong topology of  $V^*$  - uniform convergence  
on bounded sets of  $V$  (generated  
by seminorms))